Desingularization of branch points of minimal surfaces in $\mathbb{R}^4$(II)

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Abstract

We desingularize a branch point p of a minimal disk $F_0(\mathbb{D})$ in $\mathbb{R}^4$ through immersions F_t's which have only transverse double points and are branched covers of the plane tangent to $F_0(\mathbb{D})$ at p. If F_0 is a topological embedding and thus defines a knot in a sphere/cylinder around the branch point, the data of the double points of the F_t's give us a braid representation of this knot as a product of bands.

1 Introduction

1.1 The purpose

Minimal surfaces in $\mathbb{R}^4$ are immersed except at branch points, near which the surface is a N-branched covering of the tangent plane at the branch point (for some $N > 1$). In [Vi 2] we looked at a minimal map $F_0 : \mathbb{D} \longrightarrow \mathbb{R}^4$ with a branch point at the origin and we described how to desingularize F_0 through minimal immersions F_t's with only transverse double points. However, unlike F_0 these F_t's were not branched coverings of the disk. We discuss here a desingularization through immersions which are not necessary minimal but which remain N-branched coverings of the disk.

If F_0 is a topological embedding, we recall that the intersection of $F_0(\mathbb{D})$ with a small sphere (equivalently a small cylinder) centered at the branch point defines a knot which has a representation as a N-braid (cf. [S-V] mimicking the construction of [Mi]). In that case, we will use a construction of Rudolph to show how the double points of the immersions F_t's appear in a

band representation of this braid (i.e. an expression of the braid as a product of conjugates of braid generators and of their inverses).

1.2 The setting

We consider a branched immersion

$$F_0 : \mathbb{D} \longrightarrow \mathbb{R}^4 \cong \mathbb{C}^2 \times \mathbb{C}^2$$

$$F_0 : z \mapsto (z^N + h_1(z), h_2(z)) \tag{1}$$

where, for $i = 1, 2$, $h_i : \mathbb{D} \longrightarrow \mathbb{C}$ is a function with $|h_i(z)| = o(|z|^N)$.
It is standard (cf. for example [G-O-R]) to introduce a function $w : \mathbb{D} \longrightarrow \mathbb{D}$ such that

$$w(z)^N = z^N + h_1(z) \tag{2}$$

and which verifies

$$w = z + o(|z|) \qquad z = w + o(|w|) \tag{3}$$

Possibly after restricting ourselves to a smaller disk centered at 0, we reparametrize $\mathbb{D}$ with w so we can rewrite F in terms of w as

$$F_0 : w \mapsto (w^N, h(w)) \tag{4}$$

where $h(w) = o(|w|^N)$.

Remark 1. *Throughout this paper, we only use the fact that F_0 is a real analytic branched immersion, not that it is minimal. We could probably also do without the real analytic assumption.*

1.3 The construction

For λ, μ small complex numbers, we will be considering the immersions

$$F_{\lambda,\mu} : w \mapsto (w^N, h(w) + \lambda w + \mu \bar{w}) \tag{5}$$

possibly adding a small correction term if necessary:

$$F_{\lambda,\mu,\gamma} : w \mapsto \left(w^N, h(w) + \lambda w + \mu \bar{w} + Re(\gamma w^2)\right) \tag{6}$$

where γ is very small compared to λ and μ and is only introduced to give more wiggle room for transversality arguments.

Remark 2. *We used immersions similar to (5) in [Vi 1] where we established a connection between the algebraic crossing number of the braid and the normal bundle of the branched disk in an ambient 4-manifold.*

The paper is devoted to proving the following:

Theorem 1. *For λ, μ generic and small enough, $F_{\lambda,\mu}$ has a finite number of crossing points $m_1, ..., m_n$, all transverse.*
Assume that F_0 is a topological embedding and let K be the knot defined by the branch point. If $\frac{\mu}{\lambda}$ is small enough (resp. large enough), the knot K is represented by a N-braid β which is the product of the following pieces:

1. $$\prod_{2k,\ 2\leq 2k\leq N-1} \sigma_{2k}$$
   $$(\textit{resp.} \quad \prod_{2k,\ 2\leq 2k\leq N-1} \sigma_{2k}^{-1})$$

2. $$\prod_{2k+1,\ 1\leq 2k+1\leq N-1} \sigma_{2k+1}$$
   $$(\textit{resp.} \quad \prod_{2k+1,\ 1\leq 2k+1\leq N-1} \sigma_{2k+1}^{-1})$$

3. *for every double point $m_1, ..., m_n$ of $F_{\lambda,\mu}$, one copy of*
   $$b(m_i)\sigma_{k(m_i)}^{2\epsilon(m_i)}b(m_i)^{-1} \tag{7}$$
   where
   - *$\epsilon(m_i)$ is the sign of the intersection point m_i*
   - *$k(m_i) \in \{1, ..., N-1\}$*
   - *$b(m_i)$ is some element of the braid group B_N.*

1.4 Trivial knots

It follows from the expression of the braid that, if $F_{\lambda,\mu}$ is an embedding for $\frac{\lambda}{\mu}$ large enough or small enough, the knot K is trivial. There exist branched minimal disks with corresponding knots which are non trivial but have 4-genus 0, for example 10_{155} (cf. [S-V]). For such a knot, the signed number of double points of the $F_{\lambda,\mu}$ (for $\frac{\lambda}{\mu}$ large enough or small enough) is zero but $F_{\lambda,\mu}$ has necessarily double points.

1.5 Sketch of the paper

We will first establish some properties of the $F_{\lambda,\mu}$'s, for generic λ, μ's; then we will construct a closed loop Γ in the plane Π_2 generated by the first two coordinates. The braid β considered in Th. 1 will be defined as

$$\beta = \pi_2^{-1}(\Gamma) \cap F_{\lambda,\mu}(\mathbb{D}) \tag{8}$$

for λ, μ small enough and where

$$\pi_2 : \mathbb{R}^4 \longrightarrow \Pi_2 \tag{9}$$

is the orthogonal projection.

2 The family of immersions

Lemma 1. *For generic λ, μ's, the following is true:*
if $w_1, w_2 \in \mathbb{D}$ verify $w_1 \neq w_2$ and $F_{\lambda,\mu}(w_1) = F_{\lambda,\mu}(w_2)$, then the two tangent planes $dF_{\lambda,\mu}(w_1)(\mathbb{R}^2)$ and $dF_{\lambda,\mu}(w_2)(\mathbb{R}^2)$ are transverse.

Remark 3. *Lemma 1 does not exclude the possibility of triple points (i.e. three disks meeting at a point, every two of them transversally): we will see that later.*

Proof. The proof is based on the Transversality Lemma. We introduce

$$\Phi : \mathbb{C} \times \mathbb{C} \times \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{R}^4 \oplus \mathbb{R}^4$$

$$\Phi(\lambda, \mu, w_1, w_2) = (F_{\lambda,\mu}(w_1), F_{\lambda,\mu}(w_2)) \tag{10}$$

and we check that it is transverse to the diagonal Δ_8 of $\mathbb{R}^4 \oplus \mathbb{R}^4$ for $w_1 \neq w_2$. We derive from the basis (e_1, e_2, e_3, e_4) of $\mathbb{R}^4$ (in which (4) is written) a basis

$$(e_1^{(1)}, ..., e_4^{(1)}, e_1^{(2)}, ..., e_4^{(2)})$$

of $\mathbb{R}^4 \oplus \mathbb{R}^4$; thus the diagonal Δ_8 is generated by $(e_1^{(1)} + e_1^{(2)}, ..., e_4^{(1)} + e_4^{(2)})$. A point in the preimage of Δ_8 *via* Φ is of the form (λ, μ, w_1, w_2), where $w_2 = \nu w_1$ for a complex number ν verifying

$$\nu^N = 1.$$

We introduce real coordinates by setting

$$\lambda = \lambda_1 + \lambda_2 \qquad w_1 = x_1 + iy_1 \qquad w_2 = x_2 + iy_2 \tag{11}$$

and we compute the following determinant at points $w_1, w_2 = \nu w_1$; the subscripts denote the components in the basis (e_1, e_2, e_3, e_4):

$$det(\frac{\partial\Phi}{\partial x_1}, \frac{\partial\Phi}{\partial x_2}, \frac{\partial\Phi}{\partial\lambda_1}, \frac{\partial\Phi}{\partial\lambda_2}, e_1^{(1)} + e_1^{(2)}, e_2^{(1)} + e_2^{(2)}, e_3^{(1)} + e_3^{(2)}, e_4^{(1)} + e_4^{(2)}) =$$

$$\begin{vmatrix} (\frac{\partial F}{\partial x_1})_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ (\frac{\partial F}{\partial x_1})_2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ (\frac{\partial F}{\partial x_1})_3 & 0 & x_1 & -y_1 & 0 & 0 & 1 & 0 \\ (\frac{\partial F}{\partial x_1})_4 & 0 & y_1 & x_1 & 0 & 0 & 0 & 1 \\ 0 & (\frac{\partial F}{\partial x_2})_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & (\frac{\partial F}{\partial x_2})_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & (\frac{\partial F}{\partial x_2})_3 & x_2 & -y_2 & 0 & 0 & 1 & 0 \\ 0 & (\frac{\partial F}{\partial x_2})_4 & y_2 & x_2 & 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} (\frac{\partial F}{\partial x_1})_1 & -(\frac{\partial F}{\partial x_2})_1 & 0 & 0 \\ (\frac{\partial F}{\partial x_1})_2 & -(\frac{\partial F}{\partial x_2})_2 & 0 & 0 \\ (\frac{\partial F}{\partial x_1})_3 & -(\frac{\partial F}{\partial x_2})_3 & x_1 - x_2 & y_2 - y_1 \\ (\frac{\partial F}{\partial x_1})_4 & -(\frac{\partial F}{\partial x_2})_4 & y_1 - y_2 & x_1 - x_2 \end{vmatrix}$$

$$= [(x_1 - x_2)^2 + (y_1 - y_2)^2][-(\frac{\partial F}{\partial x_1})_1(\frac{\partial F}{\partial x_2})_2 + (\frac{\partial F}{\partial x_2})_1(\frac{\partial F}{\partial x_1})_2]$$

$$= |1-\nu|^2 N^2 |w|^{2N} Im(\nu). \tag{12}$$

Similarly we show

$$det(\frac{\partial\Phi}{\partial x_1}, \frac{\partial\Phi}{\partial y_2}, \frac{\partial\Phi}{\partial\lambda_1}, \frac{\partial\Phi}{\partial\lambda_2}, e_1^{(1)} + e_1^{(2)}, e_2^{(1)} + e_2^{(2)}, e_3^{(1)} + e_3^{(2)}, e_4^{(1)} + e_4^{(2)}) =$$

$$= -|1-\nu|^2 N^2 |w|^{2N} Re(\nu) \tag{13}$$

Lemma 1 follows from (12) and (13). $\square$

We denote by Π_3 the 3-plane Π_3 generated by the first 3 coordinates; we let

$$\pi_3 : \mathbb{R}^4 \longrightarrow \Pi_3 \tag{14}$$

be the orthogonal projection and we show a lemma similar to Lemma 1 for $\pi_3 \circ F_{\lambda,\mu}$:

Lemma 2. *For generic λ, μ, the following is true:*
if $w_1, w_2 \in \mathbb{D}$ verify $w_1 \neq w_2$ and $(\pi_3 \circ F_{\lambda,\mu})(w_1) = (\pi_3 \circ F_{\lambda,\mu})(w_2)$, then the two tangent planes $\pi_3\big(dF_{\lambda,\mu}(w_1)(\mathbb{R}^2)\big)$ and $\pi_3\big(dF_{\lambda,\mu}(w_2)(\mathbb{R}^2)\big)$ are transverse.

Proof. Similarly to above, we introduce the map

$$\Psi : \mathbb{C} \times \mathbb{C} \times \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{R}^3 \oplus \mathbb{R}^3$$

$$\Psi : (\lambda, \mu, w_1, w_2) \mapsto \Big((\pi_3 \circ F_{\lambda,\mu})(w_1), (\pi_3 \circ F_{\lambda,\mu})(w_2)\Big) \tag{15}$$

By truncating the determinants appearing in the proof of Lemma 1, we get

$$det(\frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial \lambda_1}, e_1^{(1)} + e_1^{(2)}, e_2^{(1)} + e_2^{(2)}, e_3^{(1)} + e_3^{(2)}) = (x_1 - x_2) N^2 |w|^{2N-2}$$

$$det(\frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}, \frac{\partial \Psi}{\partial \lambda_2}, e_1^{(1)} + e_1^{(2)}, e_2^{(1)} + e_2^{(2)}, e_3^{(1)} + e_3^{(2)}) = -(y_1 - y_2) N^2 |w|^{2N-2}$$

Hence Ψ is transverse to the diagonal Δ_6 of $\mathbb{R}^3 \oplus \mathbb{R}^3$ and Lemma 2 follows. $\square$

Next we show that $\pi_3 \circ F_{\lambda,\mu}$ has only a finite number of triple points:

Lemma 3. *Let*

$$\nu = e^{\frac{2\pi}{N} i} \tag{16}$$

and let k, l be two different integers in $\{1, ..., N-1\}$. For generic λ, μ, there is a finite number of points $w \in \mathbb{D}$ such that

$$Re\big(\lambda w + \mu \bar{w} + h(w)\big) = Re\big(\lambda \nu^k w + \mu \bar{\nu}^k \bar{w} + h(\nu^k w)\big) = Re\big(\lambda \nu^l w + \mu \bar{\nu}^l \bar{w} + h(\nu^l w)\big) \tag{17}$$

Proof. We let

$$\psi : \mathbb{C} \times \mathbb{D} \longrightarrow \mathbb{R}^2$$

$$\psi(\lambda, w) = \Big(Re\big[\lambda(1 - \nu^k) w + \mu(1 - \bar{\nu}^k)\bar{w} + h(w) - h(\nu^k w)\big],$$

$$Re[\lambda(1 - \nu^l) w + \mu(1 - \bar{\nu}^l)\bar{w} + h(w) - h(\nu^l w)]\Big).$$

We define

$$(1-\nu^k)w = w_1^{(k)} + iw_2^{(k)} \qquad (1-\nu^l)w = w_1^{(l)} + iw_2^{(l)} \tag{18}$$

and compute

$$det(\frac{\partial\psi}{\partial\lambda_1}, \frac{\partial\psi}{\partial\lambda_2}) = \begin{vmatrix} w_1^{(k)} & -w_2^{(k)} \\ & \\ w_1^{(l)} & -w_2^{(l)} \end{vmatrix} = Im[(1-\nu^k)w(1-\bar{\nu}^l)\bar{w}]$$

$$= |w|^2\Big[\sin(\frac{2\pi}{N}l) - \sin(\frac{2\pi}{N}k) + \sin\big(2\frac{\pi}{N}(k-l)\big)\Big]$$

$$= 4|w|^2 \sin(\frac{\pi}{N}l)\sin(\frac{\pi}{N}k)\sin\big(\frac{\pi}{N}(k-l)\big) \tag{19}$$

which is not zero. We use the Transversality Lemma again and conclude that for a generic λ, $\psi(\lambda,.)$ is transverse to $(0,0)$, that is, $(0,0)$ is attained at a finite number of points. $\square$

NOTATIONS. We remind the reader that π_2 is the projection onto the plane Π_2 generated by the first two coordinates.
We denote by $W_{\lambda,\mu}$ the set of w's in $\mathbb{D}\backslash\{0\}$ which verify (17) for some k, l and we let

$$X_{\lambda,\mu} = (\pi_2 \circ F_{\lambda,\mu})(W_{\lambda,\mu}) \tag{20}$$

If $D_{\lambda,\mu} \in \mathbb{R}^4$ is the set of double points of $F_{\lambda,\mu}$, we let

$$\mathcal{D}_{\lambda,\mu} = \pi_2(D_{\lambda,\mu}) \tag{21}$$

In the following lemma, we use $F_{\lambda,\mu,\gamma}$ defined in (6); nevertheless we keep the notation $\mathcal{D}_{\lambda,\mu}$ and $X_{\lambda,\mu}$ in order not to burden the notations.

Lemma 4. *For generic λ's, μ's, γ*

$$\mathcal{D}_{\lambda,\mu} \cap X_{\lambda,\mu} = \varnothing$$

In particular $F_{\lambda,\mu}$ does not have triple points (cf. Remark 3).

Proof. We pick a very small positive number η (how small it need to be will be clear from the proof below) and given a 7-uple $A = (a, b, c, d, e, \alpha, \beta) \in \mathbb{R}^7$, we define

$$H(A, w) = aRe(w) + bIm(w) + \alpha Re(w^2) + \beta Re(e^{i\eta}w^2) + i[dRe(w) + eIm(w)] \tag{22}$$

and we define

$$F(A,w) = F(w) + \big(0, H(A,w)\big) = \big(w^N, H(A,w) + h(w)\big) \tag{23}$$

We let j, k, l be three different integers in $\{1, ..., N-1\}$ and we define

$$S(A,w) = \Big(Re\big(H(A,w) - H(A, e^{\frac{2j\pi}{N}i}w)\big) + Re\big(h(w) - h(e^{\frac{2j\pi}{N}i}w)\big),$$

$$Im\big(H(A,w) - H(A, e^{\frac{2j\pi}{N}i}w)\big) + Im\big(h(w) - h(e^{\frac{2j\pi}{N}i}w)\big),$$

$$Re\big(H(A, e^{\frac{2k\pi}{N}i}w) - H(A, e^{\frac{2l\pi}{N}i}w)\big) + Re\big(h(e^{\frac{2k\pi}{N}i}w) - h(e^{\frac{2l\pi}{N}i}w)\big)\Big)$$

We show

Sublemma 1. *The map S is transverse to $(0,0,0)$; thus, for a generic A, $(0,0,0)$ is not attained by $S(A,.)$.*

Proof. We set $w = re^{i\theta}$ and we compute

$$det(\frac{\partial S}{\partial a}, \frac{\partial S}{\partial b}, \frac{\partial S}{\partial d}) = r^3\Big(\cos\theta - \cos(\theta + \frac{2j\pi}{N})\Big)\Delta = -2\sin(\frac{j\pi}{N})\sin(\theta + \frac{j\pi}{N})r^3\Delta$$

$$det(\frac{\partial S}{\partial a}, \frac{\partial S}{\partial b}, \frac{\partial S}{\partial e}) = r^3\Big(\sin\theta - \sin(\theta + \frac{2j\pi}{N})\Big)\Delta = -2\sin(\frac{j\pi}{N})\cos(\theta + \frac{j\pi}{N})r^3\Delta$$

where

$$\Delta = \begin{vmatrix} \cos\theta - \cos(\theta + \frac{2j\pi}{N}) & \sin\theta - \sin(\theta + \frac{2j\pi}{N}) \\ \cos(\theta + \frac{2k\pi}{N}) - \cos(\theta + \frac{2l\pi}{N}) & \sin(\theta + \frac{2k\pi}{N}) - \sin(\theta + \frac{2l\pi}{N}) \end{vmatrix}$$

$$= \begin{vmatrix} 2\sin(\theta + \frac{j\pi}{N})\sin(\frac{j\pi}{N}) & 2\cos(\theta + \frac{j\pi}{N})\sin(\frac{j\pi}{N}) \\ -2\sin(\theta + \frac{(k+l)\pi}{N})\sin(\frac{(k-l)\pi}{N}) & 2\cos(\theta + \frac{(k+l)\pi}{N})\sin(\frac{(k-l)\pi}{N}) \end{vmatrix}$$

$$= 4\sin(\frac{\pi}{N}j)\sin\Big(\frac{\pi}{N}(k-l)\Big)\sin\Big(\frac{\pi}{N}(j-k-l)\Big) \tag{24}$$

If (24) is zero, then

$$j = k + l \tag{25}$$

We now assume (25) and we compute

$$det(\frac{\partial S}{\partial a},\frac{\partial S}{\partial \alpha},\frac{\partial S}{\partial d}) = r^4\Big(\cos\theta - \cos(\theta+\frac{2j\pi}{N})\Big)\tilde{\Delta} = -2\sin(\frac{j\pi}{N})\sin(\theta+\frac{j\pi}{N})r^4\tilde{\Delta}$$

$$det(\frac{\partial S}{\partial a},\frac{\partial S}{\partial \alpha},\frac{\partial S}{\partial e}) = r^4\Big(\sin\theta - \sin(\theta+\frac{2j\pi}{N})\Big)\tilde{\Delta} = -2\sin(\frac{j\pi}{N})\cos(\theta+\frac{j\pi}{N})r^4\tilde{\Delta}$$

where

$$\tilde{\Delta} = \begin{vmatrix} \cos\theta - \cos(\theta+\frac{2j\pi}{N}) & \cos(2\theta) - \cos(2\theta+\frac{4j\pi}{N}) \\ \cos(\theta+\frac{2k\pi}{N}) - \cos(\theta+\frac{2l\pi}{N}) & \cos(2\theta+\frac{4k\pi}{N}) - \cos(2\theta+\frac{4l\pi}{N}) \end{vmatrix}$$

$$= \begin{vmatrix} 2\sin(\theta+\frac{j\pi}{N})\sin(\frac{j\pi}{N}) & 2\sin(2\theta+\frac{2j\pi}{N})\sin(\frac{2j\pi}{N}) \\ -2\sin(\theta+\frac{j\pi}{N})\sin(\frac{(k-l)\pi}{N}) & -2\sin(2\theta+\frac{2j\pi}{N})\sin(\frac{2(k-l)\pi}{N}) \end{vmatrix}$$

$$= 4\sin(\theta+\frac{j\pi}{N})\sin(2\theta+\frac{2j\pi}{N})\begin{vmatrix} \sin(\frac{j\pi}{N}) & \sin(\frac{2j\pi}{N}) \\ -\sin(\frac{(k-l)\pi}{N}) & -\sin(\frac{2(k-l)\pi}{N}) \end{vmatrix}$$

$$= -16\sin(\theta+\frac{j\pi}{N})\sin(2\theta+\frac{2j\pi}{N})\sin(\frac{j\pi}{N})\sin(\frac{(k-l)\pi}{N})\sin(\frac{l\pi}{N})\sin(\frac{k\pi}{N}) \quad (26)$$

The product (26) is not zero unless

$$\sin(\theta+\frac{j\pi}{N})\sin(2\theta+\frac{2j\pi}{N}) = 0$$

in which case we redo the above calculations replacing $\frac{\partial}{\partial\alpha}$ by $\frac{\partial}{\partial\beta}$ and get a non-zero determinant. This concludes the proof of Sublemma 1. $\square$

Given A, there is a unique (λ,μ,γ) such that for every w,

$$H(A,w) = \lambda w + \mu\bar{w} + Re(\gamma w^2) \quad (27)$$

Moreover the map

$$A \mapsto (\lambda,\mu,\gamma)$$

defined by (27) is a surjective submersion. Thus Lemma 4 follows from Sublemma 1. $\square$

3 The 1-complex A in Π_2

We derive from Lemma 2 that the set

$$\mathcal{A} = \{(w_1, w_2) \in \mathbb{D} \times \mathbb{D} / w_1 \neq w_2 \text{ and } \pi_3 \circ F_{\lambda,\mu}(w_1) = \pi_3 \circ F_{\lambda,\mu}(w_2)\} \quad (28)$$

is a manifold. If $(w_1, w_2) \in \mathcal{A}$, then

$$w_1^N = w_2^N$$

and we let A be the subset of $\mathbb{D} \subset \Pi_2$ consisting of the w_i^N's for (w_1, w_2) in $\mathcal{A}$. Directly by hand or by standard analytic geometry arguments (A is the projection of an anlytic set and is of dimension 1, hence it is semi-analytic and so it is stratified, see [Ło]), we derive

Lemma 5. *The set A is a 1-submanifold of $\mathbb{D}$ with a finite set of singular points which we denote $\Sigma(A)$.*

Moreover we have

Lemma 6. *The elements of $\mathcal{D}_{\lambda,\mu}$ (cf. 21) are regular points of A.*

Proof. If $p \in \mathcal{D}_{\lambda,\mu}$, there exists $w \in \mathbb{D}$ and a number ν with $\nu^N = 1$, $\nu \neq 1$ such that $p = w^N$ and

$$Re(\lambda w + \mu\bar{w} + h(w)) = Re(\lambda\nu w + \mu\bar{\nu}\bar{w} + h(\nu w)) \quad (29)$$

It follows from Lemma 4 that in a neighbourhood of p, A identifies with

$$A_\nu = \{w^N / \pi_3(F(w)) = \pi_3(F(\nu w))\}.$$

By the transversality arguments we have been using, we see that, for λ, μ generic, the set of w's which verify (29) is a 1-submanifold, hence A_ν is one too. $\square$

4 The loop Γ in $\mathbb{D}$

We now construct a closed loop Γ in $\mathbb{D}$. It stays in a small circle around the origin but leaves it to circle around the the points of $\mathcal{D}_{\lambda,\mu}$. We require Γ to always meet A transversally. The closed loop Γ splits $\mathbb{D}$ into two connected components, U_0 and U_1 and we require

- the origin 0 and the elements of $\mathcal{D}_{\lambda,\mu}$ are all in U_0
- the points of $X_{\lambda,\mu}$ (cf. 20 for the definition of $X_{\lambda,\mu}$) are all in U_1: this is possible since $X_{\lambda,\mu}$ and $\mathcal{D}_{\lambda,\mu}$ do not intersect.

We can now consider three knots in cylinders, namely

$$K = F(\mathbb{D})\cap\pi_2^{-1}(\partial\bar{\mathbb{D}}_2) \quad K_{\lambda,\mu} = F_{\lambda,\mu}(\mathbb{D})\cap\pi_2^{-1}(\partial\bar{\mathbb{D}}_2) \quad \hat{K}_{\lambda,\mu} = F_{\lambda,\mu}(\mathbb{D})\cap\pi_2^{-1}(\Gamma) \tag{30}$$

We claim that they are all isotopic. For K and $K_{\lambda,\mu}$ to be isotopic, it is enough to take λ and μ small enough.
Since there are no double points in $\pi_2^{-1}(U_1)$, the set $M_1 = F_{\lambda,\mu}(\mathbb{D})\cap\pi_2^{-1}(U_1)$ is a submanifold of $\mathbb{R}^4$; moreover, if $m \in M_1$, a vector T in Π_2 has a unique lift in $T_m M_1$. Thus, if we smoothly deform Γ to $\partial\bar{\mathbb{D}}$, we can lift this deformation into an isotopy between $K_{\lambda,\mu}$ and $\hat{K}_{\lambda,\mu}$.

4.1 Construction of Γ

It is made of three pieces:

4.1.1 The circles Γ_i's around the points in $\mathcal{D}_{\lambda,\mu}$

We let

$$\mathcal{D}_{\lambda,\mu} = \{p_1, ...p_n\} \tag{31}$$

The indexing i is chosen so that

$$arg(p_1) \geq arg(p_1) \geq ... \geq arg(p_n) \tag{32}$$

For every $i = 1, ..., n$, the point p_i is a regular point of A (cf. Lemma 6) so we can pick a small circle Γ_i in $\mathbb{D} \subset \Pi_2$ centered at p_i and such that

1. the disk bounded by Γ_i does not contain any point in $\Sigma(A)$ or a point in $\mathcal{D}_{\lambda,\mu}$ different from p_i
2. Γ_i and A meet transversally at two points:

$$\Gamma_i \cap A = \{P_i, Q_i\} \tag{33}$$

4.1.2 The circle C_ρ around the origin

We pick a small positive number ρ; we will indicate below how small we need ρ to be but for the moment we only require

$$\rho < \frac{1}{2} \inf |p_i| \tag{34}$$

and we let C_ρ be the circle in $\mathbb{D}_2$ centered at the origin and of radius ρ.

4.1.3 The $\mathcal{T}_i$'s between C_ρ and the Γ_i's

We pick a point u_i on Γ_i different from P_i, Q_i. For every i, we pick a path L_i between u_i and C_ρ and a small closed tubular neighbourhood $\mathcal{T}_i$ of L_i.
We pick the $\mathcal{T}_i$'s disjoint from one another. Moreover we require for every i that

1. $\mathcal{T}_i$ does not contain any point of $\mathcal{D}_{\lambda,\mu}$ or $\Sigma(A)$
2. $\mathcal{T}_i \cap C_\rho \cap A = \varnothing$
3. $\mathcal{T}_i \cap \Gamma_i$ does not contain P_i and Q_i
4. the boundary $\partial\mathcal{T}_i$ meets A transversally.

4.1.4 Conclusion: the loop Γ and the knot/braid $\hat{K}_\lambda$

To go along the loop Γ, we start at a point X_0 in C_ρ which does not belong to A. We follow C_ρ counterclockwise; everytime we meet a $\partial\mathcal{T}_i$, we go along it until we meet Γ_i; then we follow Γ_i till we come to the next component of $\partial\mathcal{T}_i$ which we follow back to C_ρ.

5 The crossing points of $\hat{K}_\lambda$

We now write $\hat{K}_\lambda$ as a braid β.
We denote by Π_{34} the plane in $\mathbb{R}^4$ generated by the last two coordinates. If γ is a point of Γ, there are N points $\tilde{\gamma}_1, \tilde{\gamma}_2, ..., \tilde{\gamma}_N$ in Π_{34} such that for all i,

$$(\gamma, \tilde{\gamma}_i) \in \hat{K}_\lambda$$

A crossing point $\gamma^{(0)}$ of $\hat{K}_\lambda$ is a point where the $Re(\tilde{\gamma}_i^{(0)})$'s take less than N distinct values, i.e. there are two different points $(\gamma^{(0)}, \tilde{\gamma}_i^{(0)})$ and $(\gamma^{(0)}, \tilde{\gamma}_j^{(0)})$ with the same first three coordinates. In other words, a crossing point occurs when Γ meets A.
To formalize this, we parametrize Γ as

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{D} \tag{35}$$

with $\gamma(\theta_0) = \gamma^{(0)}$. We renumber the i's and we reparametrize the $\tilde{\gamma}_i$'s in a neighbourhood of θ_0 so that

$$Re\big(\tilde{\gamma}_1(\theta_0)\big) \geq Re\big(\tilde{\gamma}_2(\theta_0)\big) \geq ... \geq Re\big(\tilde{\gamma}_k(\theta_0)\big) = Re\big(\tilde{\gamma}_{k+1}(\theta_0)\big) \geq ... \geq Re\big(\tilde{\gamma}_N(\theta_0)\big)$$

This gives us the braid generator

$$\sigma_k^{\eta(\gamma_0)} \tag{36}$$

with $\eta(\gamma_0) \in \{-1, +1\}$.
Note that if there are two different integers i, j, with $1 \leq i, j \leq N-1$ such that

$$Re\big(\tilde{\gamma}_i(\theta_0)\big) = Re\big(\tilde{\gamma}_{i+1}(\theta_0)\big) \quad \text{and} \quad Re\big(\tilde{\gamma}_j(\theta_0)\big) = Re\big(\tilde{\gamma}_{j+1}(\theta_0)\big)$$

then $|i-j| \geq 2$, which implies that the corresponding $\sigma_i^{\pm}$ and $\sigma_j^{\pm}$ commute; thus it does not matter in which order we write them in the expression of β.
The sign $\eta(\gamma_0)$ of the crossing point in (36) is the sign of

$$[Im\big(\tilde{\gamma}_{k+1}(\theta_0)\big) - Im\big(\tilde{\gamma}_k(\theta_0)\big)][Re\big(\tilde{\gamma}'_k(\theta_0)\big) - Re\big(\tilde{\gamma}'_{k+1}(\theta_0)\big)] \tag{37}$$

This sign is well-defined: the first factor in (37) is non zero, otherwise we would have a double point of $F_{\lambda,\mu}$ and we have assumed that none of the double points of $F_{\lambda,\mu}$ project to a point in Γ.
Let us see why the second factor of (37) is non-zero. The planes $\pi_3\big(T_{(\gamma(\theta_0),\tilde{\gamma}_k(\theta_0))}F_{\lambda,\mu}(\mathbb{D})\big)$ and $\pi_3\big(T_{(\gamma(\theta_0),\tilde{\gamma}_{k+1}(\theta_0))}F_{\lambda,\mu}(\mathbb{D})\big)$ are transverse (see Lemma 2) so they intersect in a line generated by a vector X which projects to a vector tangent to A. The vector $\Big(\gamma'(\theta_0), Re\big(\tilde{\gamma}'_k(\theta_0)\big)\Big)$ - resp. $\Big(\gamma'(\theta_0), Re\big(\tilde{\gamma}'_{k+1}(\theta_0)\big)\Big)$ - completes X in a basis of $\pi_3(T_{(\gamma(\theta_0),\gamma_k(\theta_0))})$ - resp. $\pi_3(T_{(\gamma(\theta_0),\gamma_{k+1}(\theta_0))})$. It follows that

$$Re\big(\tilde{\gamma}'_k(\theta_0)\big) \neq Re\big(\tilde{\gamma}'_{k+1}(\theta_0)\big)$$

and the sign (37) is well-defined.

We now examine the three types of crossing points.

5.0.5 On the circle C_ρ

We first investigate the crossing points of the braid

$$w \mapsto (w^N, \lambda w) \quad \text{(resp.} \quad w \mapsto (w^N, \mu \bar{w})) \tag{38}$$

Without loss of generality, we assume that λ and μ are real so the crossing points of the braids are given by

$$\cos \frac{2\pi}{N}(\theta + k) = \cos \frac{2\pi}{N}(\theta + l) \tag{39}$$

for $k, l \in \{1, ..., N-1\}$ and $\theta \in [\zeta, 1+\zeta]$, where ζ is a small positive number which we introduce to avoid crossing points at the endpoints of the interval. We get two values of θ for (39), namely

$$\theta_1 = \frac{1}{2}, \quad \theta_2 = 1.$$

The integers k, l appearing in (39) verify $k + l = N - 1$ (resp. $k + l = N - 2$) for θ_1 (resp. θ_2). The corresponding values for $\cos \frac{2\pi}{N}(\theta + k)$ are

$$\text{the } \cos(2s+1)\frac{\pi}{N}\text{'s with } 0 \leq 2s \leq N-2$$

$$\text{(resp. the } \cos(2s+2)\frac{\pi}{N}\text{'s with } 0 \leq 2s \leq N-3)$$

Thus the $\cos \frac{2\pi}{N}(\theta+k)$'s go through the values of $\cos \frac{\pi}{N} m$ with $1 \leq m \leq N-1$. We conclude: the crossing points above θ_1 (resp. θ_2) correspond to the braid generators $\sigma_{2k+1}^{\pm 1}$, $1 \leq 2k+1 \leq N-1$ (resp. $\sigma_{2k}^{\pm 1}$, $1 \leq 2k \leq N-1$).

It follows from (37) that a crossing point (θ_1, θ_2) of $w \mapsto (w^N, \lambda w)$ (resp. $w \mapsto (w^N, \mu \bar{w})$) is of the same sign as

$$(\sin \theta_1 - \sin \theta_2)^2 \quad \text{(resp.} \quad -(\sin \theta_1 - \sin \theta_2)^2)$$

hence they are all positive (resp. all negative).
Unlike for the braids (38) the crossing points of β on C_ρ will not all occur above the same two points of C_ρ; however, if ρ is small enough and $\frac{\lambda}{\mu}$ is large enough or small enough, the pieces in β corresponding to the crossing points of C_ρ are given by the braids (38): if that is, the crossing points of $\hat{K}$ on C_ρ translate into the two pieces of β described in 1. and 2. of Th. 1.

5.0.6 On Γ_i

We recall that m_i is the double point of $F_{\lambda,\mu}$ which projects to the center of Γ_i. There exist $w_1, w_2 \in \mathbb{D}$, $w_1 \neq w_2$ with

$$F_{\lambda,\mu}(w_1) = F_{\lambda,\mu}(w_2) = m_i.$$

We pick a neighbourhood V_1 of w_1 (resp. V_2 of w_2) in $\mathbb{D}$. We know that $\pi_3(F_{\lambda,\mu}(V_1))$ and $\pi_3(F_{\lambda,\mu}(V_2))$ intersect transversally; the curve $\pi_3(F_{\lambda,\mu}(V_1)) \cap \pi_3(F_{\lambda,\mu}(V_2))$ projects to A on Π_2. We also know that Γ_i meets A exactly at two points P_i, Q_i (cf. §4.1.1): the preimages of P_i and Q_i on $\pi_3(F_{\lambda,\mu}(V_1)) \cap \pi_3(F_{\lambda,\mu}(V_2))$ give us two braid generators $\sigma_k^{\pm}$ (with the same k).
We know from Lemma 4 that these are the only braid generators corresponding to the crossing points P_i and Q_i; to get a complete picture of the braid above Γ_i, we just need to figure out the sign of each of the two $\sigma_k^{\pm}$'s:

Lemma 7. *Let Q be one of the crossing points of β on the circle Γ_i. Let $m_i \in \mathbb{R}^4$ be the double point of $F_{\lambda,\mu}$ which projects to p_i. The sign of the crossing point Q is equal to the sign of the double point m_i as a double point of $F_{\lambda,\mu}$.*

Proof. We let T_0 and T_1 be the two tangent planes to $F_{\lambda,\mu}(\mathbb{D})$ at m_i and we construct positive bases of $\mathbb{R}^4$, T_0 and T_1.
Since the planes T_0 and T_1 intersect transversally, the planes $\pi_3(T_0)$ and $\pi_3(T_1)$ also intersect transversally. We let U be a vector in Π_3 generating $\pi_3(T_0) \cap \pi_3(T_1)$; since $\pi_2 \circ F_{\lambda,\mu}(w) = w^N$, $\pi_2 \circ F_{\lambda,\mu}$ is a local immersion outside of 0 and U projects to a non-zero vector u in Π_2 which is tangent to A.
We let v be a vector tangent to Γ at Q oriented in the direction of Γ; possibly after changing u in $-u$, (u, v) is a positive basis of Π_2 (we remind the reader that we have assumed that A and Γ meet transversally).
Because $\pi_2 \circ F_{\lambda,\mu}$ is a local immersion outside of 0, there exists a unique $u_i \in P_i$ and a unique $v_i \in P_i$ with

$$\pi_2(u_i) = u \qquad \pi_2(v_i) = v$$

Moreover $\pi_2 \circ F_\lambda$ preserves the orientation, hence the basis (u_0, v_0) (resp. (u_1, v_1)) is a positive basis of T_0 (resp. T_1).

We let (e_1, e_2, e_3, e_4) be an orthonormal positive basis of $\mathbb{R}^4$ with e_1, e_2 in Π_2 and we define another positive basis of $\mathbb{R}^4$ by

$$\mathcal{B} = (u, v, e_3, e_4). \tag{40}$$

We write the coordinates in $\mathcal{B}$ of the vectors in the bases base vectors of T_0 and T_1, namely

$$u_0 = (1, 0, \alpha, \gamma) \quad v_0 = (0, 1, \beta, \delta)$$
$$u_1 = (1, 0, \alpha, \gamma') \quad v_1 = (0, 1, \beta', \delta')$$

and we compute the determinant

$$det(u_0, v_0, u_1, v_1) = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \alpha & \beta & \alpha & \beta' \\ \gamma & \delta & \gamma' & \delta' \end{vmatrix} = -(\beta - \beta')(\gamma - \gamma') \tag{41}$$

We now recover from (41) the sign of the crossing points of the braid given by (37).
For $i = 0, 1$, we let V_i be a small disk in $F_{\lambda,\mu}(\mathbb{D})$ tangent to T_i. We denote again the two strands which meet at the crossing point by the coordinates in $\mathbb{C} \oplus \mathbb{C}$: $(\gamma(\theta), \tilde{\gamma}_k(\theta))$ and $(\gamma(\theta), \tilde{\gamma}_{k+1}(\theta))$. Locally, one is the lift of Γ to V_0 and the other one is the lift of Γ to V_1.
Since the v_i's both project to v, we derive that v_0 (resp. v_1) is the vector tangent to $F_{\lambda,\mu} \cap \pi_2^{-1}(\Gamma)$ above Q on V_0 (resp. V_1). Hence

$$Re\big(\tilde{\gamma}_k'(\theta_0)\big) - Re\big(\tilde{\gamma}_{k+1}'(\theta_0)\big)$$

has the same sign as $\beta - \beta'$.

We now use the fact that Q belongs to A. Since p_i is a regular point of A, A is parametrized near q_i by

$$a : t \mapsto q_i + tu + o(t^2) \tag{42}$$

Since (u,v) is a positive basis of Π_2 and v is tangent to Γ at Q, the point Q is on the side of the positive t's in (42). The lift of A to V_0 (resp. V_1) is parametrized by

$$\tilde{a}_0(t) = m_i + tu_0 + o(t^2) \qquad \tilde{a}_1(t) = m_i + tu_1 + o(t^2) \tag{43}$$

Thus, if we have taken Γ_i small enough, $Im(\tilde{\gamma}_k(\theta_0)) - Im(\tilde{\gamma}_{k+1}(\theta_0))$ is of the same sign as $(u_0)_4 - (u_1)_4 = \gamma - \gamma'$. $\square$

Thus the circle Γ_i contributes $\sigma_k^{2\epsilon(q_i)}$ to the braid.

5.0.7 On $\partial\mathcal{T}_i$

We proceed as in [Ru 1].
If $\mathcal{T}_i$ is a small enough neighbourhood, the map $F_{\lambda,\mu} : F_{\lambda,\mu}^{-1}(\mathcal{T}_i) \longrightarrow \mathcal{T}_i$ is a covering, hence $F_{\lambda,\mu}^{-1}(\mathcal{T}_i)$ is a disjoint union of N copies of $L_i \times [-\eta, +\eta]$ for a small $\eta > 0$.
If q_0 is a point in $L_i \cap A$, there are two points q_1 and q_2 close to q_0 in $\mathcal{T}_i \cap A$, one in each component of $\mathcal{T}_i$. If the k-th and $(k+1)$-th leaf of $\pi_3 \circ F_{\lambda,\mu}(\mathbb{D})$ coincide above q_0, the same is true for q_1 and q_2. Hence q_1 and q_2 each give us a braid generator $\sigma_k^{\pm}$ for β.
These two $\sigma_k^{\pm}$'s have opposite signs. Indeed, if we look at formula (37), the factors $Im\big(\tilde{\gamma}_{k+1}(\theta_0)\big) - Im\big(\tilde{\gamma}_k(\theta_0)\big)$ take the same sign for both q_1 and q_2, whereas the factors $Re\big(\tilde{\gamma}'_k(\theta_0)\big) - Re\big(\tilde{\gamma}'_{k+1}(\theta_0)\big)$ take opposite signs.

Putting all the $\sigma_k^{\pm 1}$'s together, we get an element $b_i \in B_N$ such that the piece of the braid which consists in going along $\mathcal{T}_i$, around Γ_i and back along $\mathcal{T}_i$ can be written as

$$b_i \sigma_{k(i)}^{2\epsilon(Q)} b_i^{-1} \tag{44}$$

where $k(i)$ is an integer in $\{1, ..., N-1\}$ and $\epsilon(Q)$ is the sign of the crossing point.
We get the terms in the braid of Th. 1 3 and the proof of Th. 1 is completed.

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